Volatility and Discount Rate

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Risk and Return

Investors are risk averse in that they prefer to invest in a ***sure thing*** rather than an ***unsure thing*** even if both investments have the same expected value. Table 1 presents three investments that have the same expected value but with differing levels of risk (i.e. variance). There are two possible states of the world, W_1 and W_2, each with probability $\frac{1}{2}$. Investment C is riskier than investment B, which is riskier than investment A. All else being equal a risk averse would demand the highest rate of return for investment C and the lowest rate of return for investment A.

Table 1: Three Investment Alternatives

Description	Probability	Inv A	Inv B	Inv C
W_1	0.50	200	250	300
W_2	0.50	200	150	100
Mean	–	200	200	200
Variance	–	0	2500	10000
Volatility (%)	–	0	25	50

For our purposes, volatility (square root of variance divided by the mean) will come from two sources, a stochastic revenue growth rate and the ratio of fixed costs to revenue. The greater the variance in revenue growth rate the greater the variance in net income. The greater the ratio of fixed costs to revenue the greater the variance in net income. The greater the level of risk, the greater will be the risk premium demanded by investors. The risk premium is a function of volatiltiy. To calculate the risk premium we need to estimate volatility and have a mechanism in place to price for it.

Why is this relevant? Imagine, if you will, that we are valuing a start-up company. In the near-term the start-up's revenue growth rate mean and volatility are very high as is the ratio of fixed costs to revenue. As the company matures it's growth rate slows, revenue becomes less volatile and fixed costs become a smaller percentage of revenue. Clearly net income is more volatile in the near-term and less volatile in the long-term. How can we pick one discount rate that would be applicable to all periods?

Note: We will assume that 100% of net income is distributed as dividends (i.e. dividends = free cash flow = net income) and that the capital structure includes zero debt (i.e. no financial leverage, only operating leverage).

Legend of Symbols

C_t	=	Contribution margin in period t
F_t	=	Fixed costs in period t
G_t	=	One plus revenue growth rate in period t
H_t	=	One plus fixed cost inflation rate in period t
N_t	=	Net income in period t
R_t	=	Revenue in period t
r_m	=	Return on the market index (S&P 500)
rf	=	Risk-free rate
s	=	Sharpe ratio
t	=	Time period number
σ_m	=	Standard deviation of market returns

Discount Rate as a Function of Volatility

The Sharpe ratio is a measure of the excess return (or Risk Premium) per unit of risk in an investment asset or a trading strategy. The Sharpe ratio is used to characterize how well the return of an asset compensates the investor for the risk taken. When comparing two assets each with the expected return against the same benchmark return, the asset with the higher Sharpe ratio gives more return for the same risk. In an efficient market with all else being equal, all investments should have the same Sharpe ratio. The historical Sharpe ratio for the S&P 500 is approximately...

$$sharpe = \frac{r_m - rf}{\sigma_m} = \frac{11.5\% - 5\%}{16\%} = 0.40 \tag{1}$$

By rearranging equation (1) we get an equation for market return that defines market return as a function of a target Sharpe ratio and market return volatility...

$$r_m = sharpe \times \sigma_m + rf \tag{2}$$

We can rearrange equation (2) to define the required rate of return (discount rate) on any asset to be a function of Sharpe ratio, expected asset return and asset return volatility. It is important to remember that the market return volatility represents the volatility of an index where diversification removes specific risk such that the only risk remaining is unspecific (systematic) risk. We will define θ to be the percent of asset return volatility that remains after diversification (percent of volatility that is systematic risk). The equation for discount rate r is therefore...

$$r = sharpe \times \sigma \times \theta + rf \tag{3}$$

Net Income Equation

We will define net income in any time period t as a function of revenue and fixed costs from the proceeding period, contribution margin, a deterministic inflation rate and a stochastic revenue growth rate. The equation for net income in any time period t is...

$$N_t \,=\, C_t\, R_{t-1} G_t \,-\, F_{t-1}\, H_t \tag{4}$$

We will assume that the contribution margin (C_t) and inflation rate (H_t) are constant and will drop the applicable time subscripts. Noting that net income in any time period is now a function of the preceding time period we can rewrite equation (4) as a function of base period ($t = 0$) revenue and fixed costs. The equation for net income in any time period t is now...

$$N_t \,=\, C\, R_0 \prod_{n=1}^{t} G_n \,-\, F_0\, H^t \tag{5}$$

Moments of the Net Income Distribution

Because revenue growth is stochastic there is a probability distribution of possible net income values for any time period t. The first and second moments of the probability distribution are the expected value of net income (N_t) and the expected value of net income squared (N_t^2).

The equation for the first moment of the probability distribution is...

$$\begin{aligned}\mathbb{E}\Big[N_t\Big] &= \mathbb{E}\Big[CR_0\prod_{i=1}^{t}G_i - F_0H^t\Big]\\ &= \mathbb{E}\Big[CR_0\prod_{i=1}^{t}G_i\Big] - \mathbb{E}\Big[F_0H^t\Big]\\ &= CR_0\,\mathbb{E}\Big[\prod_{i=1}^{t}G_i\Big] - F_0H^t \end{aligned} \tag{6}$$

The equation for the second moment of the probability distribution is...

$$\begin{aligned}\mathbb{E}\Big[N_t^2\Big] &= \mathbb{E}\Big[\Big[CR_0\prod_{i=1}^{t}G_i - F_0H^t\Big]^2\Big]\\ &= \mathbb{E}\Big[C^2R_0^2\prod_{i=1}^{t}G_i^2 - 2CR_0\prod_{i=1}^{t}G_iF_0H^t + F_0^2H^{2t}\Big]\\ &= C^2R_0^2\,\mathbb{E}\Big[\prod_{i=1}^{t}G_i^2\Big] - 2CR_0F_0H^t\,\mathbb{E}\Big[\prod_{i=1}^{t}G_i\Big] + F_0^2H^{2t} \end{aligned} \tag{7}$$

We will assume that revenue growth rate in one period is uncorrelated with revenue growth rate in another period (i.e. growth rates are independent). We can now write the expectations for growth rate and the square of growth rate (used in equations (6) and (7), respectively) as...

$$\mathbb{E}\Big[\prod_{i=1}^{t}G_i\Big] = \prod_{i=1}^{t}\mathbb{E}\Big[G_i\Big] \tag{8}$$

$$\mathbb{E}\Big[\prod_{i=1}^{t}G_i^2\Big] = \prod_{i=1}^{t}\mathbb{E}\Big[G_i^2\Big] \tag{9}$$

Net Income Distribution Mean and Variance

Net income mean (average) is expected net income in any time period t. If we were forecasting net income this is the value that we would report. The equation for net income mean (which is equation (6) above) is...

$$\begin{aligned} mean &= \mathbb{E}\Big[N_t\Big]\\ &= CR_0\prod_{i=1}^{t}\mathbb{E}\Big[G_i\Big] - F_0H^t \end{aligned} \tag{10}$$

The variance of net income is a measure of the dispersion of possible net income values around the mean. The greater the variance, the more unsure we are as to actual net income. The equation for net income variance is...

$$\begin{aligned} variance &= \mathbb{E}\Big[N_t^2\Big] - \Big[\mathbb{E}\Big[N_t\Big]\Big]^2\\ &= C^2R_0^2\prod_{i=1}^{t}\mathbb{E}\Big[G_i^2\Big] - C^2R_0^2\prod_{i=1}^{t}\mathbb{E}\Big[G_i\Big]^2\\ &= C^2R_0^2\Big[\prod_{i=1}^{t}\mathbb{E}\Big[G_i^2\Big] - \prod_{i=1}^{t}\mathbb{E}\Big[G_i\Big]^2\Big] \end{aligned} \tag{11}$$

A Hypothetical Case

Base year revenue and fixed costs are \$1,000 and \$600, respectively. Future revenue growth depends on one of three states of the world, each with probability 1/3. The state of the world in one period does not depend on prior period state of the world (i.e. independent). Fixed cost inflation rate is 3% per annum. Contribution margin is 60%.

Revenue growth factor (1 + growth rate) given state of the world W_i.

Year	W_1	W_2	W_3
1	2.00	1.50	1.20
2	1.50	1.25	1.10
3	1.20	1.10	1.00

Our results matrix is...

Row	Description	Year 1	Year 2	Year 3
1	Net income - mean	282	488	582
2	Net income - variance	60,000	130,000	166,600
3	Net income - volatility	87%	74%	70%
4	Discount rate - risk free	5.00%	10.00%	15.00%
5	Discount rate - risk premium	23.16%	19.68%	18.71%
6	Discount rate - total	28.16%	29.68%	33.71%
7	PV of cash flow	220	377	435

Where...

Row 1 - Net income mean is calculated using equation (10) above.
Row 2 - Net income variance is calculated using equation (11) above.
Row 3 - Net income volatility is square root of row 2 divided by row 1.
Row 4 - Discount rate (risk free) is risk free rate times number of years.
Row 5 - Discount rate (risk premium) is row 3 times percent systematic risk times target sharpe ratio.
Row 6 - Discount rate (total) is row 4 plus row 5 in accordance with equation (3) above.
Row 7 - PV of cash flows is row 1 divided by 1 plus row 6.